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ARTICLE

The transformation problem under positive rank one input matrices: on a new approach by Schefold

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Abstract

Recently, Schefold proposed a new approach to the transformation problem consisting of two contentions. One is that input matrices of large-scale economies can be approximated by positive rank one matrices. The second is that additional original assumptions about the relationship among input coefficients, labor coefficients, gross outputs, and surplus outputs can establish the equality of the total profits and total surplus value under a numéraire equalizing the total production prices and total value. The purpose of this study is to critically examine the second part of Schefold's argument. First, we will confirm that it is *an attempt to give plausible grounds* to a condition that has been known as sufficient for the successful solution of the transformation problem but considered to hold only by chance. Next, we will indicate that, except for a supposition about the rank of input matrices, his key assumptions depend on the measurement units of each product. Because this dependence implies that these assumptions hold only when measurement units fill particular conditions, it naturally casts a serious doubt on the generality of the analysis based on them. While it is possible to combine these assumptions into a unit-independent form, such a reformulation deprives them of the meaning attached to their original form. Thus, in spite of its unique viewpoint, Schefold's new approach does not succeed in bringing the value system closer to the production price system.

Keywords Transformation problem · Total profits · Total surplus value · Positive rank one matrix · Measurement units

JEL Classification B14 · B24 · B51

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1 Introduction

In the 1950s–1970s, the problem of the transformation from value to production prices as well as surplus value to profits in Marx's theoretical system was comprehensively clarified using Leontief's framework (Okishio 1954; Morishima 1973; Steedman 1977). One of the main results of this clarification was a demonstration of the impossibility of double aggregate equalities. In other words, if a numéraire (a bundle of products whose price is defined as one) is chosen so that the total production prices is equal to the total labor value, the total profits is generally not equal to the total surplus value. According to this result, profit cannot be regarded as redistributed surplus value without certain additional assumptions or some redefinition of core terms including "value" and "surplus value".

Recently, however, Schefold (2016) presented a set of original supplementary assumptions designed to make total profits equal to total surplus value (by presupposing the equality of total production prices and total value—hereafter we shall omit this proviso). It consists of the following four assumptions:

(i) The input matrix is positive and of rank one, thus, all of its eigenvalues other than the Frobenius root vanish.
(ii) Let $\mathbf{v}$ be the vector representing the deviations of the labor coefficient vector from the right-hand Frobenius vector of the input matrix (which Schefold calls "the Marx-vector"). Then, the sum of all the elements of this vector is equal to zero.
(iii) The covariance between the above deviation vector and the gross output vector is equal to zero.
(iv) The covariance between the above deviation vector and the surplus output vector is equal to zero.

The purpose of this study is to examine these assumptions and their consequences. To consider the purely theoretical aspects of this problem, we restrict our attention only to assumptions (ii)–(iv) and shall not argue whether input matrices can actually be approximated by positive rank one matrices. To put it differently, we are exclusively concerned with relationships that hold under positive rank one input matrices and their significance for the transformation problem. In dynamic models using the Leontief-type input matrix, the existence of its eigenvalues other than the Frobenius root sometimes becomes a source of complication.[1] If we can assume that all non-Frobenius eigenvalues are zero (or are negligibly small), then analyses of these dynamic models are considerably simplified. Therefore, it is of great interest to explore to what extent this assumption on input matrices simplifies the relationship between profits and surplus value.

[1] Foster (1963) demonstrated that negative or complex eigenvalues of the input matrix might affect dynamic stability in his analysis of the quantity adjustment process with buffer raw material inventories and static expectation of demand. For more on this point, see Shiozawa et al. (2019), Ch. 3.

We will present our discussion in the following order. Section 2 confirms several sufficient conditions for the equality between total profits and total surplus value when the input matrix is of rank one. Section 3 shows that assumptions (ii)–(iv), together with assumption (i), certainly guarantee the condition which is one of those sufficient conditions but so far has been regarded as one without any particular economic meaning. Section 4 indicates that assumptions (ii)–(iv) depend on the measurement units of each product, i.e., they hold only under measurement units satisfying certain conditions. As we will argue, this dependence casts a serious doubt on the validity of the results based on these assumptions.

Before presenting our main point, we would like to confirm an important common ground that has been established through the Sraffian critique of the labor theory of value. The following passage shows that Schefold still keeps the classical Sraffian position: "prices are … not derived from values, but derived, without having recourse to values, from the structure of production … represented by $\mathbf{A}$ and $\mathbf{l}$, and from the distribution, represented by r." Thus, even if total profit is equal to total surplus value, "The formal redundancy of surplus value remains" (Schefold 2016, p. 177). In this way, it should be noted that Schefold's new approach is not accompanied by an intention to restore the analytical supremacy of the concept of labor value that characterizes orthodox Marxian economics.

2 Conditions for the equality of the total profits and total surplus value under positive rank one input matrices

Schefold contends that an input matrix in a large-scale economy can be approximated by a positive matrix of rank one as the number of sectors approaches infinity. This contention is based on an observation of empirically estimated input–output tables as well as mathematical properties of "random matrices" (Schefold 2016, pp. 170–171).[2] As stated above, we shall not discuss the acceptability of this assumption, and will solely examine *to what extent this assumption simplifies the situation*. While the theorems Schefold presents are intended to be similar to limit theorems, he also states that these theorems are "algebraically correct if $\mathbf{A}$ [input matrix] is of rank 1" (Schefold 2016, p. 177). This assertion allows us to directly, i.e., without considering the random or stochastic nature of input coefficients, proceed to the analysis of an economy in which the input matrix is assumed to be positive and of rank one.

An input matrix of rank one is expressed as the product of a column vector $\mathbf{c} \geq 0$ and a row vector $\mathbf{f} \geq 0$, i.e., $\mathbf{A} = \mathbf{cf}$. Vector $\mathbf{f}$, which Schefold calls "the Sraffa-vector" or "the standard-vector", represents an input bundle common to all sectors. Vector $\mathbf{c}$, which he calls "the Marx-vector", represents the amounts of this bundle that each sector requires for production per unit.[3] Schefold assumes that $\mathbf{f} > 0$, $\mathbf{c} > 0$,

[2] On the concept and properties of "random system", see also Schefold (2013, pp. 1171–1179).

[3] Our notation of variables mostly follows that of Schefold. In Schefold (2013), he calls vector $\mathbf{f}$ "the composition of capital" and vector $\mathbf{c}$ "the distribution of capital over industries" (pp. 1176, 1187).

and $\mathbf{A}$ is productive, hence $0 < \mu = \mathbf{fc} < 1$. Scalar μ is the Frobenius root of $\mathbf{A}$, and it is also its only non-zero eigenvalue.[4] Vectors $\mathbf{c}$ and $\mathbf{f}$ are $\mathbf{A}$'s right-hand and left-hand Frobenius vectors, respectively.

Labor value vector $\mathbf{t}$, measured by labor amount, is defined by the value equation $\mathbf{t} = \mathbf{At} + \mathbf{l}$, where $\mathbf{l}$ is a positive column vector of labor input coefficients.[5] Under the above assumptions, the solution of this equation is given by

$$\mathbf{t} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{l} = \left(\mathbf{I} + \frac{\mathbf{cf}}{1-\mu}\right)\mathbf{l} = \frac{1}{1-\mu}(\mathbf{I} - \mathbf{B})\mathbf{l},$$

where $\mathbf{B} = \mu\mathbf{I} - \mathbf{cf}$. Since $l_i = (1-\mu)t_i + (\mathbf{Bl})_i$, the proportionality of $\mathbf{l}$ to $\mathbf{t}$ is disturbed by vector $\mathbf{Bl}$. Production price vector $\mathbf{p}$ corresponding to the uniform profit rate r and nominal wage rate w per unit labor is defined by the price equation $\mathbf{p} = (1+r)(\mathbf{Ap} + w\mathbf{l})$; here wage is assumed to be paid *before* one cycle of production. For any r such that $r < 1/\mu - 1$, vector $\mathbf{p}$ is positive and given by

$$\mathbf{p} = w(1+r)(\mathbf{I} - (1+r)\mathbf{A})^{-1}\mathbf{l} = \frac{w(1+r)}{1-\mu(1+r)}(\mathbf{I} - (1+r)\mathbf{B})\mathbf{l}$$

Let us denote the (row) vector of the gross outputs by $\mathbf{y}$ and normalize prices by $1 = \mathbf{yp}$, then we obtain

$$w = \frac{1-\mu(1+r)}{(1+r)\mathbf{y}(\mathbf{I} - (1+r)\mathbf{B})\mathbf{l}}.$$

This is the wage-profit curve when the gross output vector is taken as a numéraire. Owing to the assumption about matrix $\mathbf{A}$, the wage-profit curve corresponding to post-paid wage $(1+r)w$ is simplified up to a hyperbola. Substituting the above w into the price equation, we obtain

$$\mathbf{p} = \frac{\mathbf{l} - (1+r)\mathbf{Bl}}{\mathbf{y}(\mathbf{I} - (1+r)\mathbf{B})\mathbf{l}} = \frac{(1-\mu)\mathbf{t} - r\mathbf{Bl}}{\mathbf{y}((1-\mu)\mathbf{t} - r\mathbf{Bl})}.$$

Let $\mathbf{y} = \mathbf{f}/(\mathbf{fl})$, then these are simplified into

$$w(1+r) = \frac{R-r}{1+R}, \quad \mathbf{p} = \mathbf{l} - (1+r)\mathbf{Bl},$$

respectively, where $R = 1/\mu - 1$ is the supremum profit rate corresponding to $w = 0$.[6] Thus, if prices are normalized by (a constant multiple of) the Sraffa-vector, production price vector $\mathbf{p}$ and post-paid wage $w(1+r)$ become linear functions of profit rate r. Likewise, let $\mathbf{l} = \mathbf{c}/(\mathbf{yc})$, then the wage-profit curve takes the same form as

4 Let n be the number of sectors, then the characteristic equation of matrix $\mathbf{A} = \mathbf{cf}$ is given by $\lambda^{n-1}(\lambda - \mu) = 0$.

5 Here we disregard problems related to heterogeneous labor, joint production, fixed capital, and choice of techniques and so on. None of them is relevance to our argument below.

6 If wage is assumed to be paid *after* production and prices are normalized by net-product vector, then we have Sraffa's famous formula $w = 1 - r/R$ (Sraffa 1960).

the case $\mathbf{y} = \mathbf{f}/(\mathbf{fl})$, and price vector $\mathbf{p}$ is always equal to $\mathbf{l} = (1-\mu)\mathbf{t}$ irrespective of profit rate r. Thus, if the labor vector is proportional to the Marx-vector, production prices are fixed to (a constant multiple of) the labor vector.

Let $\mathbf{p}_0$ denote the production prices corresponding to $r = 0$, i.e.,

$$\mathbf{p}_0 = \frac{\mathbf{l} - \mathbf{Bl}}{\mathbf{yl} - \mathbf{yBl}} = \frac{\mathbf{t}}{\mathbf{yt}}$$

This is the labor value vector normalized by the gross output vector.

Net product vector $\mathbf{y}(\mathbf{I} - \mathbf{A}) = \mathbf{y}(\mathbf{I} - \mathbf{cf})$ is divided into real wage (worker's consumption) vector $\mathbf{b}$ and surplus product vector $\mathbf{s}$, i.e., $\mathbf{y} = \mathbf{yA} + \mathbf{b} + \mathbf{s}$. So long as workers spend all of their wage income, wage rate w must satisfy $w\mathbf{yl} = \mathbf{bp}$. Schefold interprets this equation as the equilibrium condition determining wage rate w and profit rate r (Schefold 2016, p. 175). However, If the real wage bundle per unit labor is fixed to vector $\mathbf{d}$, then $w\mathbf{yl} = \mathbf{bp}$ automatically holds because $w = \mathbf{dp}$ and $\mathbf{b} = (\mathbf{yl})\mathbf{d}$.

The total profits and total surplus value are respectively given by

$$P = \mathbf{sp} = \frac{\mathbf{s}((1-\mu)\mathbf{t} - r\mathbf{Bl})}{\mathbf{y}((1-\mu)\mathbf{t} - r\mathbf{Bl})}, \quad M = \mathbf{sp}_0 = \frac{\mathbf{st}}{\mathbf{yt}},$$

where P and M represents Profit and Mehrwert, respectively. Equation $\mathbf{yp} = \mathbf{yp}_0$ always holds by the choice of numéraire (both $\mathbf{yp}$ and $\mathbf{yp}_0$ are equal to 1). On the other hand, since

$$P - M = \mathbf{sp} - \mathbf{sp}_0 = \frac{r((\mathbf{st})\mathbf{y} - (\mathbf{yt})\mathbf{s})\mathbf{Bl}}{\mathbf{y}((1-\mu)\mathbf{t} - r\mathbf{Bl})\mathbf{yt}},$$

equation $P = M$ holds if and only if $r((\mathbf{st})\mathbf{y} - (\mathbf{yt})\mathbf{s})\mathbf{Bl} = 0$. $\mathbf{Bl} = 0$ implies that $\mathbf{l}$ is the right eigenvector of matrix $\mathbf{cf}$ corresponding to eigenvalue μ, i.e., $\mathbf{l}$ is proportional to $\mathbf{c}$. Similarly, $(\mathbf{st})\mathbf{y} - (\mathbf{yt})\mathbf{s} = 0$ implies $\mathbf{s}$ is proportional to $\mathbf{y}$. Therefore, either of the following conditions is sufficient for $P = M$.

(i) $r = 0$. This is the case of non-capitalist production.
(ii) $\mathbf{l}$ is proportional to $\mathbf{c}$ (hence $\mathbf{Bl} = 0$). This is the case of uniform organic compositions.
(iii) $\mathbf{s}$ is proportional to $\mathbf{y}$ (hence $(\mathbf{st})\mathbf{y} - (\mathbf{yt})\mathbf{s} = 0$). The sufficiency of this condition is obvious from $\mathbf{yp} = \mathbf{yp}_0$. Note that it is not assured simply by the proportionality of $\mathbf{y}$ to the Sraffa-vector $\mathbf{f}$. In fact, if $\mathbf{y} = k\mathbf{f} (k > 0)$ but $\mathbf{b}$ is *not* proportional to $\mathbf{f}$, then $(\mathbf{st})\mathbf{y} - (\mathbf{yt})\mathbf{s} = k\mathbf{b}((\mathbf{ft})\mathbf{I} - \mathbf{tf}) \neq 0$. So, the condition for prices to be expressed as linear functions of profit rate is different from the condition for total profits to be equal to total surplus value.
(iv) $\mathbf{yBl} = \mathbf{sBl} = 0$. As we shall see in the next section, this is the case to which Schefold pays attention.

These are mutually independent. For example, conditions (iv) can be satisfied even if conditions (i)–(iii) are all unsatisfied. In other words, it is compatible with $r > 0$, $\mathbf{Bl} \neq 0$, and $(\mathbf{st})\mathbf{y} - (\mathbf{yt})\mathbf{s} \neq 0$.

Remarkably, none of these sufficient conditions are specific to the case of positive rank one input matrices. The only effect of this assumption is that $\mathbf{A}$'s right-hand Frobenius vector is directly expressed by $\mathbf{c}$. Thus, in spite of seemingly drastic simplification, the assumption that the input matrix is positive and of rank one, by itself, does not ease the attainment of $P = M$.[7]

3 Additional assumptions on labor coefficients and outputs

For the purpose of establishing the equality of total profits and total surplus value, Schefold further introduces several new assumptions. In those assumptions, the following two kinds of vectors play the central role. One is row vector $\mathbf{m}$, which is defined by

$$\mathbf{m} = \mathbf{yB}/\mu = \mathbf{y}(\mathbf{I} - \mathbf{cf}/\mu).$$

Vector $\mathbf{m}$ measures "the deviations between the activity levels [i.e., vector $\mathbf{y}$] and the standard vector [$\mathbf{f}$]" (Schefold 2016, p. 172). The second is column vector $\mathbf{v}$, which is defined by

$$\mathbf{v} = \mathbf{Bl}/\mu = (\mathbf{I} - \mathbf{cf}/\mu)\mathbf{l}.$$

Vector $\mathbf{v}$ measures "the deviations of the labor vector [$\mathbf{l}$] from the Marx-vector [$\mathbf{c}$]".[8] If $\mathbf{l}$ is proportional to $\mathbf{c}$, then $\mathbf{v} = 0$ and "price would be equal to labor values at all rates of profits" (Schefold 2016, p. 173). By definition, $\mathbf{v}$ is orthogonal to $\mathbf{f}$ and $\mathbf{m}$ is orthogonal to $\mathbf{c}$. In other words,

$$\mathbf{fv} = \mathbf{f}(\mathbf{I} - \mathbf{cf}/\mu)\mathbf{l} = 0, \quad \mathbf{mc} = \mathbf{y}(\mathbf{I} - \mathbf{cf}/\mu)\mathbf{c} = 0.$$

Accordingly, $\mathbf{Av} = \mathbf{c}(\mathbf{fv}) = 0$, $\mathbf{mA} = (\mathbf{mc})\mathbf{f} = 0$; i.e., $\mathbf{v}$ is one of $\mathbf{A}$'s right-hand eigenvectors corresponding to eigenvalue 0, and $\mathbf{m}$ is one of A's left-hand eigenvectors corresponding to eigenvalue 0. Hence,

$$\mathbf{sv} = (\mathbf{y} - \mathbf{yA} - \mathbf{b})\mathbf{v} = \mathbf{yv} - \mathbf{bv}.$$

Observing $\mathbf{B}^2 = \mu\mathbf{B}$, we have

$$\mathbf{mv} = \mathbf{yBl}/\mu = \mathbf{yv} = \mathbf{ml} = \mathbf{m}(\mathbf{At} + \mathbf{l}) = \mathbf{mt}.$$

Together with the assumption about input matrix, Schefold's new approach to the transformation problem consists of the following assumptions:

(i) $\mathbf{A} = \mathbf{cf}$, where both column vector $\mathbf{c}$ and row vector $\mathbf{f}$ are positive.

[7] Under general input matrices, $P = M$ holds if and only if $(\mathbf{yt})(\mathbf{sq}) - (\mathbf{st})(\mathbf{yq}) = 0$, where $\mathbf{q} = (\mathbf{I} - (1 + r)\mathbf{A})^{-1}\mathbf{l} = \mathbf{p}/((1 + r)w)$. In order for $\mathbf{q}$ to be proportional to $\mathbf{p}$ for $r > 0$, $\mathbf{l}$ must be $\mathbf{A}$'s right-hand Frobenius vector. If $w = \mathbf{dp}$, then $P = M$ is equivalent to $\mathbf{y}(\mathbf{A} + \mathbf{ld})(\mathbf{p} - \mathbf{p}_0) = 0$. This can be satisfied even if $\mathbf{y}(\mathbf{A} + \mathbf{ld})$ is not proportional to $\mathbf{y}$ so long as $\mathbf{y}(\mathbf{A} + \mathbf{ld})$ is orthogonal to $\mathbf{p} - \mathbf{p}_0$.

[8] Schefold defines vector $\mathbf{m}$ by $\mathbf{m} = \mathbf{y} - \mathbf{q}_1 = \mathbf{q}_2 + \cdots + \mathbf{q}_n$, where $\mathbf{q}_1$ is $\mathbf{A}$'s left-hand Frobenius vector and $\mathbf{q}_2, \ldots, \mathbf{q}_n$ are other eigen vectors corresponding to eigenvalue 0 (Schefold 2016, p. 172). Since $\mathbf{yA} = \mathbf{q}_1\mathbf{A} = \mu\mathbf{q}_1$, we have $\mathbf{q}_1 = (\mathbf{yc}/\mu)\mathbf{f}$, $\mathbf{m} = \mathbf{y} - (\mathbf{yc}/\mu)\mathbf{f} = \mathbf{yB}/\mu$. In a similar manner, $\mathbf{v} = \mathbf{Bl}/\mu$ is derived from $\mathbf{v} = \mathbf{l} - \mathbf{x}_1 = \mathbf{x}_2 + \cdots + \mathbf{x}_n$.

(ii) $\bar{v} = \mathbf{ev}/n = 0$, where $\mathbf{e} = [1, \dots, 1]$ and n denotes the number of sectors.
(iii) $\mathrm{cov}\,(\mathbf{m}, \mathbf{v}) = 0$, where $\mathrm{cov}\,(\mathbf{m}, \mathbf{v}) = (\mathbf{m} - \bar{m}\mathbf{e})\left(\mathbf{v} - \bar{v}\mathbf{e}^{\mathrm{T}}\right) = \mathbf{mv}/n - \bar{m}\bar{v}$, $\bar{m} = \mathbf{me}^{\mathrm{T}}/n$, and T denotes transposition.[9]
(iv) $\mathrm{cov}\,(\mathbf{s}, \mathbf{v}) = 0$, where $\mathrm{cov}\,(\mathbf{s}, \mathbf{v}) = \mathbf{sv}/n - \bar{s}\bar{v}$, $\bar{s} = \mathbf{se}^{\mathrm{T}}/n$.

Assumption $\bar{v} = 0$, which Schefold considers as "a new assumption … in the literature on Sraffa and Marx", means that "on average the deviations of the labor vector from the Marx-vector disappear". If $\mathbf{A} = \mathbf{ce}$, i.e., the common input bundle includes exactly one unit of each product, then $n\bar{v} = \mathbf{e}(\mathbf{I} - \mathbf{ce}/(\mathbf{ec}))\mathbf{l} = 0$. Accordingly, $\bar{v} = 0$ is "tendentially implied" when $\mathbf{A}$ is "random" in a sense that it approaches to $\mathbf{A} = \mathbf{ce}$ as the number of sectors increases infinitely (Schefold 2016, p. 174).

$\mathrm{cov}\,(\mathbf{m}, \mathbf{v})$ represents the sample covariance between $m_1, \dots, m_n$ and $v_1, \dots, v_n$. Schefold justifies the assumption $\mathrm{cov}\,(\mathbf{m}, \mathbf{v}) = 0$ on the ground that "there is in fact no reasons why the deviations of activities from the average industry and the deviation of the labor vector from the Marx vector are correlated" (Schefold 2016, p. 173). If both m_i's and v_i's can be regarded as samples taken from corresponding populations, each of which follows its own probability distribution, then $\mathrm{cov}\,(\mathbf{m}, \mathbf{v})$ would approach to 0 as n increases infinitely.[10] This supposition is disputable, because $m_1, \dots, m_n$ represent amounts of qualitatively different (or at least regarded to be so) commodities. However, at present we are concerned only with the consequences derived from this assumption. Since $\mathrm{cov}\,(\mathbf{m}, \mathbf{v}) = 0$ is equivalent to $\mathbf{mv} = n\bar{m}\bar{v}$, assumptions (ii) and (iii) lead to $\mathbf{yv} = \mathbf{mv} = 0$.

As to assumption (iv), Schefold gives the same explanation as the case of assumption (iii) and derives $\mathbf{sv} = 0$ from $\mathbf{sv} = n\bar{s}\bar{v}$.

In sum, assumptions (i)–(iv) give $\mathbf{yv} = \mathbf{sv} = 0$, or, $\mathbf{yBl} = \mathbf{sBl} = 0$, so we have

$$\mathbf{p} = \frac{\mathbf{l} - (1 + r)\mathbf{Bl}}{\mathbf{yl}} = \frac{(1 - \mu)\mathbf{t} - r\mathbf{Bl}}{\mathbf{y}(1 - \mu)\mathbf{t}}, \quad P = \mathbf{sp} = \frac{\mathbf{st}}{\mathbf{yt}} = \mathbf{sp}_0 = M.$$

Consequently, Schefold's additional assumptions about vectors $\mathbf{m}$ and $\mathbf{v}$ certainly guarantee $P = M$. Schefold regards this as "a most surprising result, obtained after 120 years of discussions of the transformation problem" (Schefold 2016, p. 176). Since $\mathbf{sv} = (\mathbf{y} - \mathbf{b})\mathbf{v}$, $\mathbf{yv} = \mathbf{sv} = 0$ implies $\mathbf{bv} = 0$. Thus, if $\mathbf{yv} = \mathbf{sv} = 0$, then "the vectors $\mathbf{s}$, $\mathbf{b}$, and $\mathbf{yA}$ are all in the same hyper plane, being orthogonal to $\mathbf{v}$". At the same time, however, they are "not necessarily proportional" (Schefold 2016, p. 198). In other words, condition (iv) does not imply condition (ii) nor (iii).

Substituting $\mathbf{Bl} = \mu\mathbf{v}$, condition (iv) in the previous section can be expressed as $\mathbf{yv} = \mathbf{sv} = 0$. Although the sufficiency of this condition is well-known, it has been considered as the one without particular economic meaning. In its relation to the preceding discussions on this subject, Schefold's approach is an attempt to give this condition a certain new interpretation.

[9] This definition of $\mathrm{cov}\,(\mathbf{m}, \mathbf{v})$ is given in Schefold (2013, p. 1172).

[10] In Schefold (2013, p. 1172), he explicitly supposes that "the components of the $\mathbf{m}$ and $\mathbf{v}$ as *independent random variables with small means*" (emphasis in original), and derives $\mathrm{cov}\,(\mathbf{m}, \mathbf{v}) = 0$ from this supposition.

Schefold says "Whether or not $\bar{\mathrm{v}} = 0$, prices are linear functions of the rate of profit in systems for which all non-dominant eigenvalues vanish" (Schefold 2016, p. 175). However, assumptions (i) and (iii) give only $\mathbf{p} = (\mathbf{l} - (1+r)\mathbf{Bl})/(\mathbf{yl} - (1+r)\mu\bar{\mathrm{m}}\bar{\mathrm{v}})$. In order for vector $\mathbf{p}$ to be expressed as a linear function of profit rate r, assumption (ii), $\bar{\mathrm{v}} = 0$, is indispensable.[11]

4 Dependence of critical assumptions on the choice of measurement units

An amount of a product is measured by some physically or customarily defined unit. In choosing the measurement unit of a product, there always exists a wide range of arbitrariness.[12] Besides, the choices of the measurement units of different products can be made independently from each other. Thus, *any sound economic analysis requires that the assumptions therein do not depend on the choice of measurement units of each product.* Let us examine whether Schefold's assumptions satisfy this requirement.

If current k_i units of product i is redifined as its new measurement unit, then quantities measured by product i are divided by k_i and quantities defined per unit of product i are multiplied by k_i. Hence, input coefficient a_{ij} is replaced by $k_i a_{ij}/k_j$, and thus, input matrix $\mathbf{A} = \mathbf{cf}$ is replaced by $\mathbf{A}' = \mathbf{KcfK}^{-1}$, where $\mathbf{K}$ is a diagonal matrix of $k_1, \ldots, k_n > 0$. Similarly, labor coefficient vector $\mathbf{l}$, gross product vector $\mathbf{y}$, and surplus product vector $\mathbf{s}$ are replaced by $\mathbf{l}' = \mathbf{Kl}$, $\mathbf{y}' = \mathbf{yK}^{-1}$ and $\mathbf{s}' = \mathbf{sK}^{-1}$, respectively. Let $\mathbf{c}' = \mathbf{Kc}, \mathbf{f}' = \mathbf{fK}^{-1}$, then

$$\mathbf{c}' > 0, \quad \mathbf{f}' > 0, \quad \mathbf{A}' = \mathbf{c}'\mathbf{f}', \quad \mu' = \mathbf{f}'\mathbf{c}' = \mathbf{fK}^{-1}\mathbf{Kc} = \mathbf{fc} = \mu.$$

Thus, the rank of $\mathbf{A}$ and its eigenvalues are not affected by the choice of measurement units. Since $\mathbf{B}' = \mu'\mathbf{I} - \mathbf{c}'\mathbf{f}' = \mu\mathbf{I} - \mathbf{KcfK}^{-1}$, we have

$$\begin{aligned}
\mathbf{B}'\mathbf{l}' &= \left(\mu\mathbf{I} - \mathbf{KcfK}^{-1}\right)\mathbf{Kl} = \mathbf{KBl}, \\
\mathbf{y}'\mathbf{B}' &= \mathbf{yK}^{-1}\left(\mu\mathbf{I} - \mathbf{KcfK}^{-1}\right) = \mathbf{yBK}^{-1}, \\
\mathbf{v}' &= \mathbf{B}'\mathbf{l}'/\mu' = \mathbf{KBl}/\mu = \mathbf{Kv}, \quad \mathbf{m}' = \mathbf{y}'\mathbf{B}'/\mu' = \mathbf{yBK}^{-1}/\mu = \mathbf{yK}^{-1}, \\
\mathbf{y}'\mathbf{v}' &= \mathbf{yK}^{-1}\mathbf{Kv} = \mathbf{yv}, \quad \mathbf{s}'\mathbf{v}' = \mathbf{sK}^{-1}\mathbf{Kv} = \mathbf{sv}.
\end{aligned}$$

Consequently, scalars $\mathbf{yv}(= \mathbf{mv})$ and $\mathbf{sv}$ are both independent from the choice of measurement units. In other words, if $\mathbf{yv} = 0$ and $\mathbf{sv} = 0$ hold under a certain set of measurement units, they always hold under any other sets. However, this cannot be said of vectors $\mathbf{v}$ and $\mathbf{m}$. They change as unit-replacement matrix $\mathbf{K}$ changes.

[11] Note that Schefold does not assume $\bar{\mathrm{m}} = 0$. In fact, he emphasizes that "our solution of the transformation problem should not be confused with that relying on standard proportions. It is therefore essential that the deviations $\mathbf{m}$ are not assumed to vanish" (Schefold 2016, p. 172).

[12] Measurement units of products should not be confused with units of time, distance, weight etc. When a product is measured by its length (like the case of sewing thread), the adoption of the metric system does not determine one unit of a particular kind of thread.

From the above expressions of $\mathbf{v}'$ and $\mathbf{m}'$, we have

$$\begin{aligned} n\bar{\mathrm{v}}' &= \mathbf{ev}' = \mathbf{eKv} = k_1 v_1 + \cdots + k_n v_n, \\ n\bar{\mathrm{m}}' &= \mathbf{m}'\mathbf{e}^{\mathrm{T}} = \mathbf{mK}^{-1}\mathbf{e}^{\mathrm{T}}, \quad n\bar{\mathrm{s}}' = \mathbf{s}'\mathbf{e}^{\mathrm{T}} = \mathbf{sK}^{-1}\mathbf{e}^{\mathrm{T}}, \\ \operatorname{cov}\left(\mathbf{m}', \mathbf{v}'\right) &= \mathbf{mv}/n - \bar{\mathrm{m}}'\bar{\mathrm{v}}' = \mathbf{mv}/n - \mathbf{mK}^{-1}\mathbf{e}^{\mathrm{T}}\mathbf{eKv}/n^2, \\ \operatorname{cov}\left(\mathbf{s}', \mathbf{v}'\right) &= \mathbf{sv}/n - \bar{\mathrm{s}}'\bar{\mathrm{v}}' = \mathbf{sv}/n - \mathbf{sK}^{-1}\mathbf{e}^{\mathrm{T}}\mathbf{eKv}/n^2. \end{aligned}$$

Consequently, $\bar{v}'$, cov $\left(\mathbf{m}', \mathbf{v}'\right)$ and cov $\left(\mathbf{s}', \mathbf{v}'\right)$ are all depend on $k_1, \ldots, k_n$. Thus, *they generally change when measurement units of products are changed.* Let us confirm this as to the case $n = 2$ and $k_1 = 1, k_2 = k \neq 1$. In this case,

$$\begin{aligned} \mathbf{ev} &= \left(f_2 - f_1\right)\left(c_2 l_1 - c_1 l_2\right)/\mu, \quad \mathbf{ev}' = \left(f_2 - f_1 k\right)\left(c_2 l_1 - c_1 l_2\right)/\mu. \\ \operatorname{cov}\left(\mathbf{m}', \mathbf{v}'\right) &= \mathbf{mv}/n - \left(m_1 v_1 + k m_1 v_2 + m_2 v_1/k + m_2 v_2\right)/n^2, \\ \operatorname{cov}\left(\mathbf{s}', \mathbf{v}'\right) &= \mathbf{sv}/n - \left(s_1 v_1 + k s_1 v_2 + s_2 v_1/k + s_2 v_2\right)/n^2, \end{aligned}$$

where $\mu = f_1 c_1 + f_2 c_2$. Assume $c_2 l_1 \neq c_1 l_2$, then $\mathbf{ev} \neq \mathbf{ev}'$. Thus, if either of $\bar{\mathrm{v}}$ and $\bar{\mathrm{v}}'$ is zero, then the other is not zero. As to cov $\left(\mathbf{m}', \mathbf{v}'\right)$ and cov $\left(\mathbf{s}', \mathbf{v}'\right)$, they are both hyperbolic functions with respect to k and increase infinitely as k approaches to 0 or increase infinitely.

In general, $\bar{\mathrm{v}}' = 0$ holds if and only if vector $\left[k_1, \ldots, k_n\right]$ is orthogonal to vector $\mathbf{v}$. Let us recall that $\mathbf{f}$ is one of vectors satisfying this condition. If $k_1 = f_1, \ldots, k_n = f_n$, then input matrix $\mathbf{A} = \mathbf{cf}$ is replaced by

$$\mathbf{A}' = \mathbf{FcfF}^{-1} = \mathbf{c}'\mathbf{e}, \quad \mathbf{c}' = \mathbf{Fc},$$

where $\mathbf{F}$ is a diagonal matrix of $f_1, \ldots, f_n$. As we have already shown, vector $\mathbf{v}'$ corresponding matrix $\mathbf{A}'$ satisfies $\bar{\mathrm{v}}' = \mathbf{ev}'/n = 0$. Hence, so long as $\mathbf{f} > 0$, it is always possible to choose a set of measurement units under which the simple average of the deviations of the labor vector from the Marx-vector disappear. In this sense, we may say that assumption $\mathbf{f} > 0$ already implies $\mathbf{A} = \mathbf{ce}, \bar{\mathrm{v}} = 0$. Similarly, if $k_1 = 1/c_1, \ldots, k_n = 1/c_n$, then input matrix $\mathbf{A} = \mathbf{cf}$ is replaced by

$$\mathbf{A}' = \mathbf{CcfC}^{-1} = \mathbf{ef}', \quad \mathbf{f}' = \mathbf{fC}^{-1},$$

where $\mathbf{C}$ is a diagonal matrix of $c_1, \ldots, c_n$. Vector $\mathbf{m}'$ corresponding this $\mathbf{A}'$ satisfies $\bar{\mathrm{m}}' = \mathbf{m}'\mathbf{e}^{\mathrm{T}}/n = 0$.[13]

Note that $\mathbf{f} > 0$ is itself a very strong assumption because it implies that all products are used as inputs in any sector. If some of $\mathbf{f}$'s elements are zero, then such a

[13] Although Schefold thinks "it seems more plausible that $\bar{\mathrm{v}}$ tends to zero than $\bar{\mathrm{m}}$" (2013, p. 1172), as we have shown in the text, under a positive rank one input matrix, it is always possible to choose a set of measurement units satisfying $\bar{\mathrm{m}} = 0$.

replacement is impossible. This case really occurs when some kinds of products are used only for final consumption.[14]

As we have already indicated, while $\bar{\text{v}}$, cov $(\mathbf{m}, \mathbf{v})$ and cov $(\mathbf{s}, \mathbf{v})$ are all dependent on the choice of measurement units, both $\mathbf{yv} = \mathbf{mv} = \text{cov}\,(\mathbf{m}, \mathbf{v}) + n\bar{\text{m}}\bar{\text{v}}$ and $\mathbf{sv} = \text{cov}\,(\mathbf{s}, \mathbf{v}) + n\bar{\text{s}}\bar{\text{v}}$ are unit-independent. Thus, as far as we follow the rule that assumptions should not be unit-dependent, Schefold's assumptions (ii)–(iv) must be replaced by $\mathbf{mv} = \mathbf{yv} = 0$ and $\mathbf{sv} = 0$. This means to *directly assume* condition (iv) in Sect. 2. Although Schefold derives them from assumptions (ii)–(iv), $\mathbf{mv} = 0$ implies cov $(\mathbf{m}, \mathbf{v}) = 0$ *only when the measurement units are selected to satisfy* $\bar{\text{v}} = 0$ (or $\bar{\text{m}} = 0$). In general, $\mathbf{mv} = 0$ implies cov $(\mathbf{m}, \mathbf{v}) = -n\bar{\text{m}}\bar{\text{v}}$, hence $\mathbf{mv} = 0$ is incompatible with cov $(\mathbf{m}, \mathbf{v}) = 0$ if $\bar{\text{v}} \neq 0$. The same thing can be said of the relationship between $\mathbf{sv}$ and cov $(\mathbf{s}, \mathbf{v})$. Since measurement units of products are chosen independently of each other, components of vector $\mathbf{v}$ (or vector $\mathbf{m}$) cannot be regarded as a set of samples from independent and identically distributed random variables.

As for condition (iv) ($\mathbf{yv} = \mathbf{sv} = 0$), it would be difficult to find grounds to justify this, even in the sense of a limit theorem which tendentially holds as the number of sectors increases. There seems to be no reason that both the gross and surplus product vectors must be orthogonal to vector $\mathbf{v}$, which is determined only by technological conditions. A set of the input coefficient matrix and the labor coefficient vector cannot restrict the ranges of gross and surplus products into a particular hyper plane irrespective of conditions concerning income distribution and demand for products.

5 Conclusions

Even if there are good reasons to adopt a very strong assumption that the input matrix is positive and of rank one (and thus all the non-Frobenius eigenvalues vanish), this supposition does not substantially simplify the relationship between profits and surplus value. Although this result seems unexpected, it is in truth rather a matter of course as the uniformity of the composition of input coefficients does not eliminate the inter-sectoral differences in the ratio of input coefficients to labor coefficients. After all, these differences are the most fundamental cause of the deviations of total profits from total surplus value.

Assumptions (ii)–(iv) ($\bar{\text{v}} = 0$, cov $(\mathbf{m}, \mathbf{v}) = 0$, and cov $(\mathbf{s}, \mathbf{v}) = 0$) are introduced to proceed beyond this limit of simplification. Certainly, they ensure $\mathbf{yv} = \mathbf{sv} = 0$, one of the sufficient conditions for $P = M$. However, since $\bar{\text{v}}$, cov $(\mathbf{m}, \mathbf{v})$, and cov $(\mathbf{s}, \mathbf{v})$ all depend on measurements units, any assumptions assigning them particular values lack the generality necessary for solid economic analysis. While condition $\mathbf{yv} = \mathbf{sv} = 0$ is unit-independent, it is difficult to find any persuasive grounds

[14] Both Marx and Sraffa paid great attention to the difference of positions occupied by each product (commodity) in the input–output structure. In the case of Sraffa, this concern is reflected in his concept of basic and non-basic commodities (Sraffa 1960). Assumption $\mathbf{f} > 0$ implies that there is no such difference of positions. In this economy, products can be different only in that how much they are demanded for final consumption.

for assuming them directly. If the satisfaction of this condition should be regarded as purely accidental, we need not correct the established proposition that $P = M$ cannot hold except in some special cases.

Compliance with ethical standards

Conflict of interest The author declares that he/she has no conflict of interest.

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